Warm-Up

CST 9.0

Which of the following sentences is true about the graphs of $y = 3(x-5)^2 + 1$ and $y = 3(x+5)^2 + 1$?

A Their vertices are maximums.

B The graphs have the same shape with different vertices.

C The graphs have different shapes with different vertices.

D One graph has a vertex that is a maximum, while the other graph has a vertex that is a minimum.

Current 8.0

Solve by completing the square:

$$3x^2 + 5x - 2 = 0$$

Current 9.0

Explain how the graphs for the functions $y = x^2 - 4$ and $y = (x+3)^2 - 4$ are the same and how they are different.

Current 10.0

Find the zeros for the functions $y = x^2 - 4$ and $y = (x+3)^2 - 4$. How do the zeros compare?

Solving Quadratic Equations by Completing the Square when $a \neq 1$

We can use properties of equality to transform a quadratic equation in the form $ax^2 + bx = -c$ so that the left side is a perfect square trinomial, then solve by taking the square root of both sides.

Recall that a perfect square trinomial is in the form $a^2 \pm 2ab + b^2$.
The factors of $a^2 + 2ab + b^2$ are $(a+b)(a+b)$.

This product is half of $2ab$

Examples: Factor by completing the square.

1) $x^2 + 5x = 7$

$2x^2 + 10x = 14$ ← Multiply by 2 to make the linear coefficient even.

$4x^2 + 20x = 28$ ← Multiply by 2 again to make the 1st term a square

$4x^2 + 20x + \_\_ = 28 + \_\_$

$(2x+5)(2x+5) = 28 + \underline{25}$ ← Determine what to add to make a perfect square trinomial. (determine what the factors must be)

Product must be half of 20.
Think: 2(?) = 10

$$(2x+5)^2 = 53$$

$$2x + 5 = \pm\sqrt{53}$$

$$2x = -5 \pm \sqrt{53}$$

$$x = \frac{-5 \pm \sqrt{53}}{2}$$

2)

$$3x^2 - 7x = 20$$

$$6x^2 - 14x = 40$$

$$36x^2 - 84x = 240$$

Ask students what first steps we can take to transform the left side into a perfect square trinomial:
Multiply by 2 to make the linear coefficient even,
then multiply by 6 to make the 1st term a square

$$36x^2 - 84x + \_\_ = 240 + \_\_$$

$$(6x-7)(6x-7) = 240 + \underline{49}$$

Ask what we should add to make a perfect square trinomial.
(It helps to determine what the factors must be)

Product must be half of –84
Think: 6(?) = –42

$$(6x-7)^2 = 289$$

$$6x - 7 = \pm\sqrt{289}$$

$$6x - 7 = \pm 17$$

$$6x = 7 \pm 17$$

$$x = \frac{7 \pm 17}{6}$$

$$x = 4 \text{ or } x = -\frac{5}{3}$$

3)

$$5x^2 - 3x - 4 = 0$$

$$5x^2 - 3x = 4$$ ← Rewrite in the form $ax^2 + bx = -c$

$$10x^2 - 6x = 8$$ ← Multiply by 2 to make the linear coefficient even

$$100x^2 - 60x = 80$$ ← Multiply by 10 to make the 1st term a square

$$100x^2 - 60x + \_\_ = 80 + \_\_$$

$$(10x-3)(10x-3) = 80 + \underline{9}$$

Determine what to add to make a perfect square trinomial.
(determine what the factors must be)

Product must be half of –60
Think:10(?) = –30

$$(10x-3)^2 = 89$$

$$10x - 3 = \pm\sqrt{89}$$

$$10x = 3 \pm \sqrt{89}$$

$$x = \frac{3 \pm \sqrt{89}}{10}$$

4) **You Try:** (Warm up problem)

$$3x^2 + 5x - 2 = 0$$
$$3x^2 + 5x = 2$$
$$6x^2 + 10x = 4$$
$$36x^2 + 60x = 24$$
$$36x^2 + 60x + \underline{\quad} = 24 + \underline{\quad}$$
$$(6x+5)(6x+5) = 24 + \underline{25}$$
$$(6x+5)^2 = 49$$
$$6x + 5 = \pm 7$$
$$6x = -5 \pm 7$$
$$x = \frac{-5 \pm 7}{6}$$
$$x = \frac{1}{3} \text{ or } x = -2$$

5) Solve using the method shown above.

$$ax^2 + bx + c = 0$$
$$ax^2 + bx = -c$$
$$2ax^2 + 2bx = -2c$$
← Multiply by 2 to make the linear coefficient even
$$4a^2x^2 + 4abx = -4ac$$
← Multiply by $2a$ to make the 1st term a square
$$4a^2x^2 + 4abx + \underline{\quad} = -4ac + \underline{\quad}$$
$$(2ax+b)(2ax+b) = -4ac + \underline{b^2}$$
← Determine what to add to make a perfect square trinomial. (determine what the factors must be)

Product must be half of $4ab$
Think: $2a(?) = 2ab$

$$(2ax+b)^2 = -4ac + b^2$$
$$2ax + b = \pm\sqrt{-4ac + b^2}$$
$$2ax = -b \pm \sqrt{-4ac + b^2}$$
$$x = \frac{-b \pm \sqrt{-4ac + b^2}}{2a}$$

Using u-substitution to Solve Quadratic Equations

Examples:

1) Solve: $x^2 - 4x - 5 = 0$

$a = 1,\ b = -4,\ c = -5$

Let $x = u - \frac{b}{2a}$

$x = u + \frac{4}{2(1)}$

$x = u + 2$

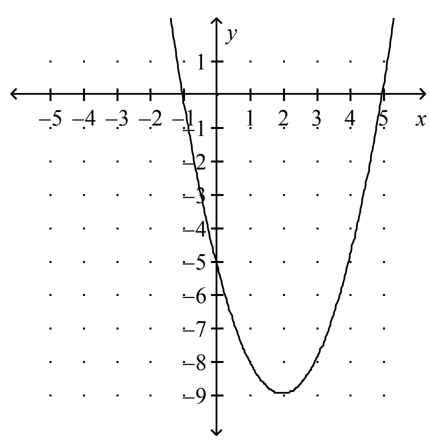
Substitute into the equation, then solve for u.

$$(u+2)^2 - 4(u+2) - 5 = 0$$
$$u^2 + 4u + 4 - 4u - 8 - 5 = 0$$
$$u^2 - 9 = 0$$
$$u^2 = 9$$
$$u = \pm 3$$

Now substitute back:

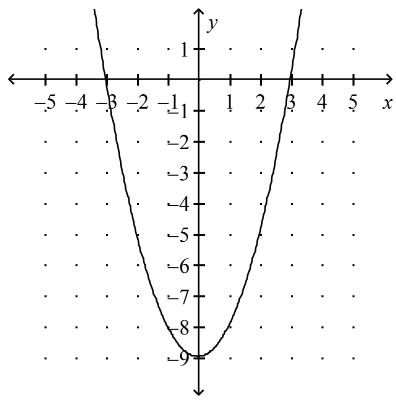
$$x = u + 2$$
$$x = \pm 3 + 2$$
$$x = -3 + 2 \text{ or } x = 3 + 2$$
$$x = -1 \text{ or } x = 5$$

Why this works: Look at the graph of the related function: $y = x^2 - 4x - 5$



The vertex of this function is (2, –9). Translate the graph horizontally so that the vertex is on the y-axis:

$T(x, y) \rightarrow (x-2, y)$



The translated function is $y = x^2 - 9$, and the zeros of this translated function are –3 and 3. Translating back by adding 2 gives the zeros of the original function: –1 and 5.

Replacing x with $u - \frac{b}{2a}$ translates the graph horizontally so that the vertex is on the y-axis.

Substituting the values for u back into $x = u - \frac{b}{2a}$ adds back the horizontal distance translated:

$u = x + \frac{b}{2a}$

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2) **You try:** Solve: $x^2 + 4x + 12 = 0$

$a = 1,\ b = 4,\ c = 12$

Let $x = u - \frac{b}{2a}$

$x = u - \frac{4}{2(1)}$

$x = u - 2$

Substitute into the equation, then solve for u.

$$(u-2)^2 + 4(u-2) + 12 = 0$$
$$u^2 - 4u + 4 + 4u - 8 + 12 = 0$$
$$u^2 + 8 = 0$$
$$u^2 = -8$$
$$u = \pm\sqrt{-8}$$
$$u = \pm i\sqrt{8}$$
$$u = \pm 2i\sqrt{2}$$

Now substitute back:

$$x = u - 2$$
$$x = -2 + u$$
$$x = -2 \pm 2i\sqrt{2}$$

3) Solve: $2x^2 - 8x + 5 = 0$

$a = 2,\ b = -8,\ c = 5$

Let $x = u - \dfrac{b}{2a}$

$$x = u + \frac{8}{2(2)}$$
$$x = u + 2$$

Substitute into the equation, then solve for u.

$$2(u+2)^2 - 8(u+2) + 5 = 0$$
$$2\left(u^2 + 4u + 4\right) - 8(u+2) + 5 = 0$$
$$2u^2 + 8u + 8 - 8u - 16 + 5 = 0$$
$$2u^2 - 3 = 0$$
$$2u^2 = 3$$
$$u^2 = \frac{3}{2}$$
$$u = \pm\sqrt{\frac{3}{2}}$$
$$u = \pm\frac{\sqrt{6}}{2}$$

Now substitute back:

$$x = u + 2$$
$$x = 2 \pm \frac{\sqrt{6}}{2}$$
$$x = \frac{4}{2} \pm \frac{\sqrt{6}}{2}$$
$$x = \frac{4 \pm \sqrt{6}}{2}$$

In general, to solve for $ax^2+bx+c=0$, let $x=u-\frac{b}{2a}$:

$$a\left(u-\frac{b}{2a}\right)^2+b\left(u-\frac{b}{2a}\right)+c=0$$

$$a\left(u^2-\frac{bu}{a}+\frac{b^2}{4a^2}\right)+b\left(u-\frac{b}{2a}\right)+c=0$$

$$au^2-bu+\frac{b^2}{4a}+bu-\frac{b^2}{2a}+c=0$$

$$au^2+\frac{b^2}{4a}-\frac{b^2}{2a}+c=0$$

$$au^2+\frac{b^2}{4a}-\frac{2b^2}{4a}+\frac{4ac}{4a}=0$$

$$au^2+\frac{-b^2+4ac}{4a}=0$$

$$au^2=\frac{b^2-4ac}{4a}$$

$$u^2=\frac{b^2-4ac}{4a^2}$$

$$u=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$u=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

Substitute back:

$$x=u-\frac{b}{2a}$$

$$x=-\frac{b}{2a}+u$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$